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OPTIMAL NUMERICAL METHOD FOR SIMULATING DYNAMIC FLOW OF GAS IN PIPELINES

A. OSIADACZ

Department of Control Engineering, Institute of Oil and Gas, Warsaw, Poland

SUMMARY

Finite difference methods for solving the linear model describing unsteady state flow in pipelines are considered in the present paper. These methods are compared with each other in order to determine the best one, which meets the criteria of accuracy and relatively small computation time.

KEY WORDS Gas Networks Computational Methods Simulation Optimization

1. INTRODUCTION

Gas plays an extremely significant role in the fuel-energetic balance of most industrialized countries of the world. High calorific value combined with the facility of transport places it in the group of most valuable row materials.

For that very reason its economic utilization is a problem of major importance. It should be dealt with by the optimization (with regard to a given criterion) of both the process of on-line control of gas transport system and the design of new or the reconstruction of existing networks. One cannot properly realize any of the enumerated tasks without first solving problems raised by network simulation.

In the process of system control the simulation supplies us with the information on the values of pressures and flows indispensable in the selection of suitable parameters both for compressor stations and reduction stations.

In the process of design the simulation allows us to correctly select network configurations, geometrical dimensions of pipelines, as well as the sites of both compressor and reduction stations for given parameters of gas supply and demand.

Two kinds of simulation are commonly differentiated: the static one and the dynamic one. The present article deals with the dynamic simulation, i.e. with the case in which the parameters characterizing the gas supply of the system and its load are functions of time (in the static simulation they are independent of time).

Correct simulation of dynamic properties necessitates the selection of the suitable mathematical model and the suitable numerical method enabling us to solve this model. Finite difference methods for solving the model elaborated in Reference 1 are considered in the present paper. These methods are compared with each other in order to determine the best one, which meets the criteria of accuracy and relatively small computation time. The investigations described have been undertaken chiefly because in many professional publications (e.g. References 2–4) various numerical schemes had been advanced, whereas the criteria for their selection had not been presented.

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2. MATHEMATICAL MODEL FOR GAS PIPELINES

An explicit model of a dynamic, physical, 'real-world' system, such as gas flowing through a pipeline, is a set of partial differential equations based upon the principles of conservation of mass and momentum together with the equation of state. While developing the explicit mathematical model for the dynamics of gas flow through a pipeline the following assumptions have been made:

(*a*) the flow is turbulent
(*b*) the gas process is isothermal
(*c*) the pipeline is rectilinear.

These assumptions are used by Charnyi[5] in describing the gas dynamics in a pipeline explicitly by means of the following set of non-linear partial differential equations:

$$\begin{cases} -\dfrac{\partial p}{\partial x} = \dfrac{\partial(\rho w)}{\partial t} + \dfrac{\lambda}{2D}\rho w^2 + \dfrac{\partial}{\partial x}((1+\beta)\rho w^2) + \rho g \sin\alpha \\ -\dfrac{\partial p}{\partial t} = c^2 \dfrac{\partial(\rho w)}{\partial x} \end{cases} \tag{1}$$

where

c = speed of sound in gas, m/s (it depends upon the gas chemical constitution and temperature)
$w = w(x, t)$ = average gas velocity (averaged over the cross-section) in the pipeline, m/s
$\rho = \rho(x, t)$ = average density (averaged over the cross-section) in the pipeline, kg/m^3
$g = 9{\cdot}81$ = acceleration due to gravity, m/s^2
$\alpha(^\circ)$ = angle of pipeline inclination with respect to the horizontal plane
λ = friction coefficient for the fluid in the pipeline
D = pipeline diameter, m
$p = p(x, t)$ = average gas pressure (averaged over the cross-section) in the pipeline, MPa
β = correction of Coriolis allowing for a profile of non-uniform velocities in the stream.

The constituent factors $\partial(\rho w)/\partial t$, $\lambda \rho w^2/2D$ and $\rho g \sin\alpha$ define the gas inertia, friction force and force of gravity, respectively. The factor $(1+\beta)\rho w^2$ is determined by the flowing gas dynamic pressure.

An optimum mathematical model for gas pipelines in real working conditions has been evaluated. The investigations[1] have shown that the pipeline dynamic should be defined by means of a linear partial differential equation with respect to p^2:

$$\frac{\partial^2(p^2)}{\partial x^2} = \frac{\lambda Q_V}{DFc^2}\frac{\partial p^2}{\partial t} \tag{2}$$

where:

F = cross-section area, m^2
Q_v = volume flow, m^3/s

Because it was assumed that $Q_v(x, t)$ would be averaged along the pipe for each time interval Δt, equation (2) is linear with respect to the square of pressure for each interval of approximation.

Under normal conditions of pipeline operation this equation was shown to be the best compromise between an accurate physical description and a representation requiring small computation time as compared to other analysed models of pipelines.

3. DESCRIPTION OF THE FINITE-DIFFERENCE SCHEMES

We have solved equation (2) by the following finite-difference schemes;

A. *Two time levels*

$$\frac{P_k^{n+1}-P_k^n}{\Delta t}=a\left(\nu\frac{P_{k-1}^{n+1}-2P_k^{n+1}+P_{k+1}^{n+1}}{(\Delta x)^2}+(1-\nu)\cdot\frac{P_{k-1}^n-2P_k^n+P_{k+1}^n}{(\Delta x)^2}\right)+O(\Delta t,(\Delta x)^2) \quad (3)$$

where

$$O\leqslant\nu\leqslant 1,\qquad a=\frac{DFc^2}{\lambda Q_v},\qquad P_k^n=p^2(x_k,t_n)$$

and Δx is the quantization element.

In particular, for $\nu=0{\cdot}5$; 1 we have, respectively:

A1. *Implicit scheme.* (Figure 1), unrestricted stability with error

$$\varepsilon=O((\Delta t)^2,(\Delta x)^2)$$

A2. *Implicit scheme.* (Figure 2), unrestricted stability with error

$$\varepsilon=O(\Delta t,(\Delta x)^2)$$

B. *Three time levels*

When the derivatives $\partial P/\partial t$ and $\partial^2P/\partial x^2$ are approximated by the equations

$$\frac{\partial P}{\partial t}=\frac{P_k^{n+1}-P_k^{n-1}}{2\Delta t}+O(\Delta t)^2 \quad (4)$$

$$\frac{\partial^2 P}{\partial x^2}=\frac{P_{k-1}^n-2P_k^n+P_{k+1}^n}{(\Delta x)^2}+O(\Delta x)^2 \quad (5)$$

and using the following formula

$$P_k^n=\frac{P_k^{n+1}+P_k^{n-1}}{2}$$

we obtain the finite-difference approximation to (2)

$$\frac{P_k^{n+1}-P_k^{n-1}}{2\Delta t}=a\frac{P_{k-1}^n-P_k^{n+1}-P_k^{n-1}+P_{k+1}^n}{(\Delta x)^2} \quad (6)$$

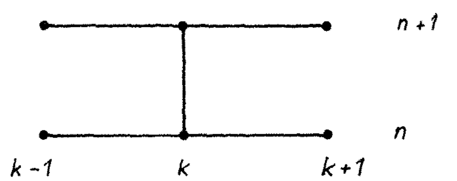


Figure 1. Crank–Nicholson implicit scheme, unrestricted stability

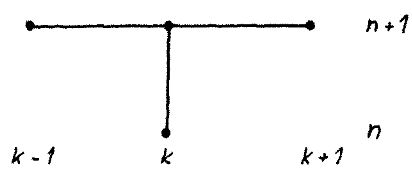


Figure 2. Implicit scheme, unrestricted stability

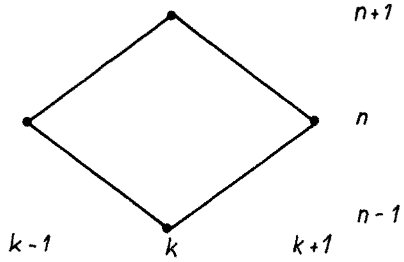


Figure 3. Dufort–Frankel explicit scheme, unrestricted stability

The Dufort and Frankel explicit scheme (equation (6), Figure 3) provides unrestricted stability with error

$$\varepsilon = O(\Delta t, (\Delta x)^2, (\Delta t/\Delta x)^2)$$

The Dufort–Frankel method is consistent with equation (2) with truncation error going to zero, only if $(\Delta t/\Delta x) \to 0$ as both Δx and $\Delta t \to 0$. Therefore, even though this method is unconditionally stable, $\Delta t \ll \Delta x$ is required for consistency. Note that, if Δt and $\Delta x \to 0$ in such a way that $\Delta t/\Delta x = c$, a constant, then the finite difference equation (6) is consistent with the following hyperbolic equation

$$\frac{\partial p^2}{\partial t} = a\left(\frac{\partial^2 p^2}{\partial x^2} - c^2 \frac{\partial p^2}{\partial t^2}\right) \tag{7}$$

Therefore, when using equation (6) in order to approximately solve equation (2) we should make use of grids with sufficiently small $c = \Delta t/\Delta x$.

Another implicit method used to solve the one dimensional diffusion equation is shown in Figure 4. It is a scheme of unrestricted stability with error,

$$\varepsilon = O(\Delta t, (\Delta x)^2)$$

Using the scheme shown in Figure 4 we transform equation (2) into

$$1{\cdot}5\,\frac{P_k^{n+1} - P_k^n}{\Delta t} - 0{\cdot}5\,\frac{P_k^n - P_k^{n-1}}{\Delta t} = a\,\frac{P_{k-1}^{n+1} - 2P_k^{n+1} + P_{k+1}^{n+1}}{(\Delta x)^2} + O(\Delta t, (\Delta x)^2) \tag{8}$$

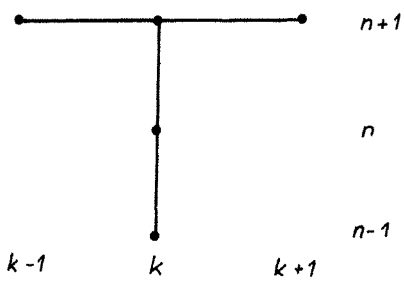


Figure 4. Three-time level implicit scheme, unrestricted stability

The schemes shown in Figures 1–4 are presented in accordance with fundamental schemes for solving parabolic equations of the following type:[6]

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2} \qquad (\sigma = \text{const} > 0) \tag{9}$$

For the implicit schemes, the unknown values of **P** at any time level are found by solving a set of algebraic equations:

$$\mathbf{A}\mathbf{P}^{n+1} = \mathbf{b} \tag{10}$$

Equations (10) take a tridiagonal form (elements occur on the main diagonal and on one subdiagonal above and below). This system of equations was solved using the Thomas algorithm.[7]

4. BOUNDARY CONDITIONS

Initial conditions are determined on the basis of the pipeline state at the moment preceding the start time of the simulation. Let us assume that the initial moment is $t = 0$. Thus Q_v^*†$(x, 0) = f(x)$, the initial flow and $p(x, 0) = h(x)$, the initial pressure profile. It was assumed that $Q_v^*(x, 0) = Q_{v,0}^* = \text{constant}$ (steady-state flow). The steady-state flow of gas under NPT conditions is given by Weymouth's equation:

$$Q_{v,0}^* = 389640 D^{8/3} \sqrt{\left(\frac{p_0^2 - p_N^2}{LTSZ}\right)} \tag{11}$$

where

L = length of pipeline, m
p_0 = gas pressure at point $x = 0$, MPa
p_N = gas pressure at point $x = L$, MPa
T = temperature, °K
Z = compression coefficient
S = specific gravity of gas

The pressure along the pipeline for the steady-state conditions of gas flow may be determined from the following relation:

$$p(x, 0) = \sqrt{\left[p^2(0, 0) - (p^2(0, 0) - p^2(L, 0))\frac{x}{L}\right]} \tag{12}$$

Boundary conditions are determined by the way in which the pipeline is supplied and loaded. It was assumed, for discussion purposes, that the pipeline section under investigation was supplied from a compressor station at $x = 0$, and fed at $x = L$ a receiver having a load $Q_v^*(t)$ varying in time. To determine the form of function $Q_v^*(t)$ at point $x = L$ of the pipeline, statistical analysis was performed for the 24-hours reports covering the period of a year (the reports included the values of pressures and flow rates for the selected points in the pipeline system). Next, the most probable flow change at the pipeline end was defined. The solution, $Q_v^*(L, t)$, is a discrete period function with a discrete step $\Delta t = 2$ h, the time interval being

† * Indicates under NPT conditions.

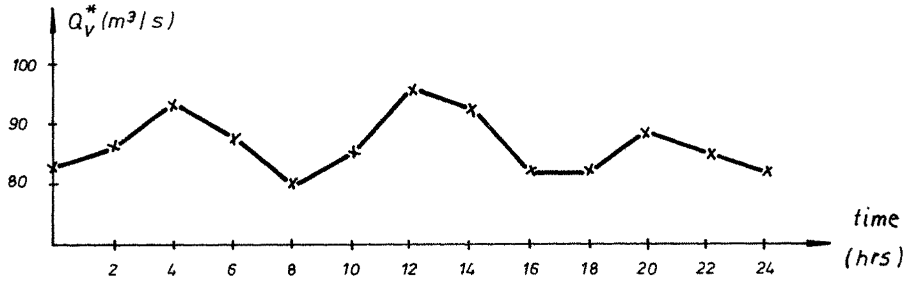


Figure 5. Changes of flow with time (boundary condition)

$t \in [0, 24\text{ h}]$ (Figure 5). It is assumed that introducing appropriate changes of the capacity of the compressor station the value of the pressure at the beginning of the pipeline may be kept invariant. Thus

$$p(0, t) = p_0 = \text{constant} \tag{13}$$

$$Q_v^*(L, t) = f(t) \tag{14}$$

The relation

$$Q_{v,N}^{*2} = (p_{N-1}^2 - p_N^2)\psi \tag{15}$$

where

$$\psi = \frac{(389640 D^{8/3})^2}{\Delta x TSZ}$$

is true only for the steady-state flow along the last discrete section Δx of the pipeline, being used for manipulating condition (14) into the form necessary for solving equation (2).

5. INVESTIGATION RESULTS

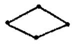
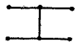
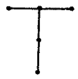
Investigations were carried out for the schemes shown in Figures 1–4. The following were taken for calculation:

$$p_0 = 4{\cdot}92 \text{ MPa}$$
$$Q_{v,0}^* = 83 \text{ m}^3/\text{s}$$
$$L = 10^5 \text{ m}$$
$$D = 0{\cdot}6 \text{ m}$$
$$N = 15$$

The assumed values of $Q_{v,0}^*$ and p_0 are the most probable ones obtained, as in the case of $Q_v^*(L, t)$, on the basis of the statistical analysis of 24-hours reports for a selected pipeline section.

The calculations were made by means of a CDC 7600 computer. The simulation period equal to 48 h was taken. The initial investigations verified that this period is ample enough for the model to 'forget' the initial condition. The computation times and accuracy were determined for every numerical scheme. The investigations have been carried out for $\Delta t = 5$ s, 300 s, 600 s, 1200 s.

Table I

	Errors '1' MPa			
Δt				
300	0·041	0·041	0·037	0·037
600	0.086	0·057	0·052	0·052
1200	0·096	0·087	0·081	0·081

The accuracy of the results for $\Delta t = 300$, 600 and 1200 has been estimated by comparing the results with those for the accurate solutions obtained for $\Delta t = 5$ s.

The results of investigations are shown in Tables I and II.

Errors '1' in Table I are absolute maximum errors and errors '2' in Table II are mean absolute errors for pressure values at discrete points.

Investigations have shown that the explicit Frankel–Dufort scheme is faster than implicit schemes as far as the computation time is concerned.

Maximum savings of computation time are 15 per cent with respect to implicit schemes, of which the scheme in Figure 2 is the fastest. The analysis of results has proved that, if Δt increases, the accuracy of the methods decreases.

Next, the relationship between the value of Δx and the efficiency of the computations was investigated. The results corresponding to $N = 10$, 7, 4 ($\Delta t = 5$ s) were compared with those corresponding to $N = 15$, $\Delta t = 5$ s. The results are shown in Tables III and Table IV.

Table II

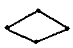
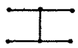
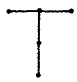
	Errors '2' MPa			
Δt				
300	0·0040	0·0039	0·0041	0·0041
600	0·0210	0·0137	0·0114	0·0120
1200	0·0238	0·0193	0·0177	0·0179

Table III

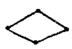
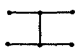
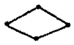
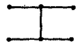
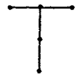
	Erros '1' MPa			
N				
4	0·063	0·058	0·059	0·059
7	0·037	0·032	0·031	0·032
10	0·021	0·018	0·019	0·019

Table IV

Errors '2' MPa				
N				
4	0·025	0·024	0·021	0·022
7	0·016	0·015	0·012	0·013
10	0·008	0·007	0·007	0·007

The correct numerical process of simulation of a gas pipeline system should provide a compromise between the computation time and the accuracy of the solution. It means that the values of Δx and Δt should be selected in such a way that the error of a numerical solution is of the same order as the error of parameters which are measured on the object (pressure, flow, temperature) and are used as the input data for the simulation program. Error-free choice of Δx and Δt should be preceded by the analysis of measurement errors.

An analysis of the results of investigation has shown that for $N=4$ and $\Delta t=600$ s the maximum numerical error was less than 3 per cent ($L=10^5$ m, $Q_v^*(L,t)$ as in Figure 5). Owing to the lack of the information necessary for the analysis of error, it was assumed that $N=4$ and $\Delta t=600$ s for the next investigations.

The simulation of gas network necessitates the numerical solution of equation (2) for each pipe. The length of pipes may change from several hundred meters to more than one hundred kilometers. In order to compare the consistency between the Dufort–Frankel scheme and the implicit schemes of Figures 1–4 investigations were made for:

$$p(0,t)=\text{constant}=4{\cdot}92\ \text{MPa}$$

$$N=4, \qquad \Delta t=600\ \text{s}$$

$$L=10^3\ \text{m} \quad \text{and} \quad L=7\times10^4\ \text{m}$$

and $Q_v^*(L,t)$ changing according to the function shown in Figure 6.

It was assumed that $Q_{v,0}^*=97{\cdot}222(\text{m}^3/\text{s})$ for $L=7\times10^4$ m and $Q_{v,0}^*=694{\cdot}444(\text{m}^3/\text{s})$ for $L=10^3$ m.

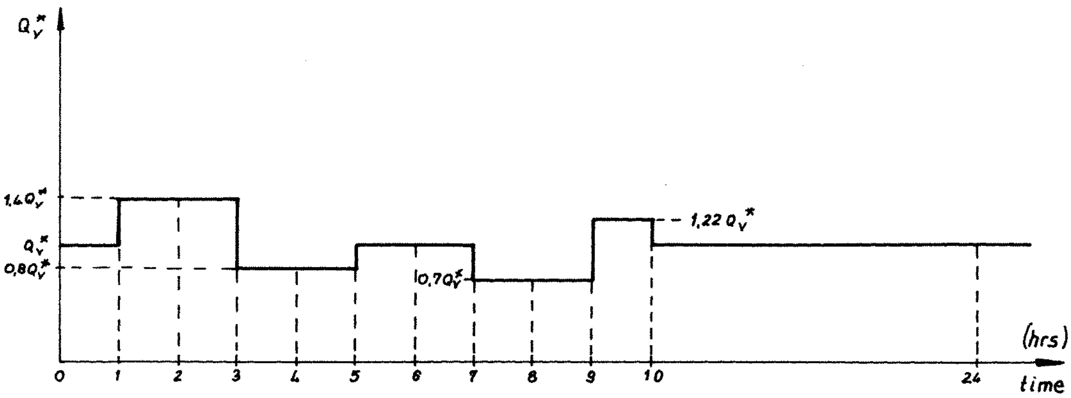


Figure 6. Changes of flow with time (boundary condition)

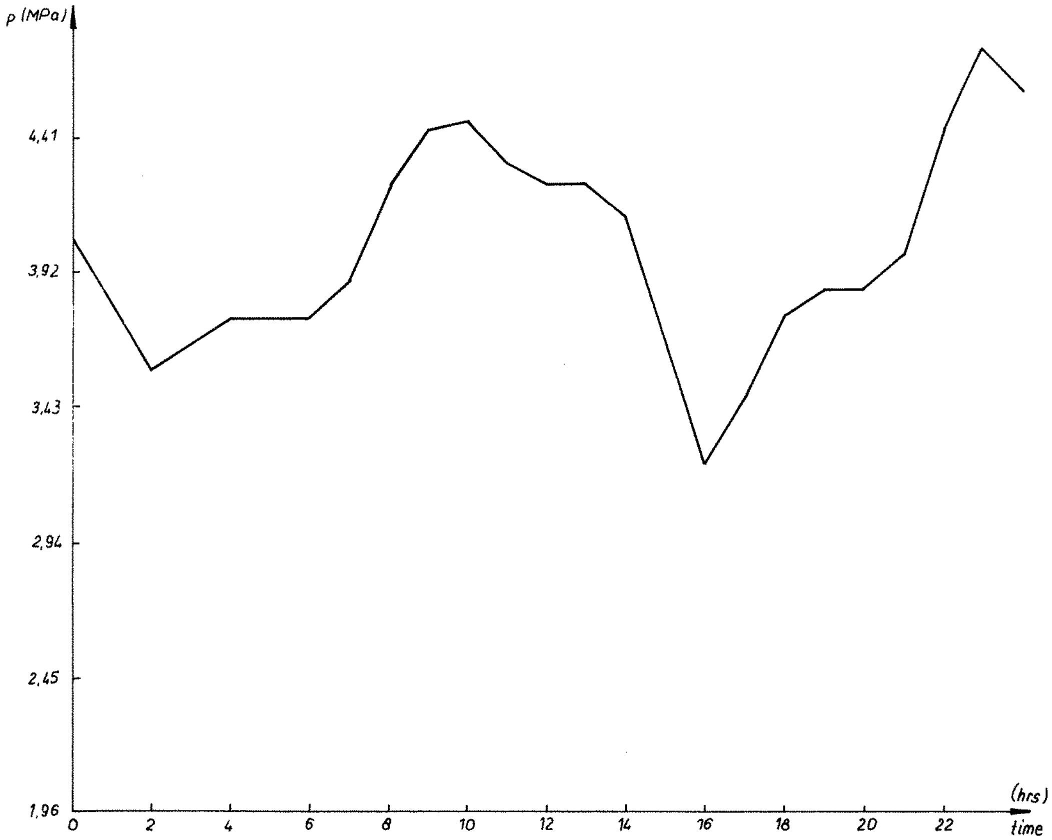


Figure 7. Changes of pressure at $x = L$ obtained using D-F scheme for $L = 10^3$ m

The investigation results for pressure changes at $x = L$ obtained using the Dufort–Frankel scheme for $L = 10^3$ m and $L = 7 \times 10^4$ m are shown in Figures 7 and 8, respectively.

This example shows that for a short pipe the Dufort–Frankel method is not consistent. It is possible to improve its consistency by decreasing the value of Δt or increasing N, but in both cases it leads to an increase of computation time. For the implicit schemes the results are in both cases practically the same (the differences are smaller than 2 per cent).

The comparison of computation times given by implicit schemes has shown that the scheme presented in Figure 2 is the fastest (0·039 s) with respect to 0·04 s for the scheme shown in Figure 1 and 0·042 s for the scheme in Figure 4 (CDC-7600 computer).

6. CONCLUSIONS

The investigations have shown that the pipeline dynamics should be solved numerically by means of the implicit schemes. The Dufort–Frankel method is accurate only for $c \ll 1 (c = \Delta t/\Delta x)$. This means that in the case of simulating a network of a complex structure when the number of discretization intervals is the same for every pipeline, the value of Δt should be selected in such a way as to ensure the correct solution for shortest pipelines. This significantly increases the overall simulation time. One may partially counteract the effect

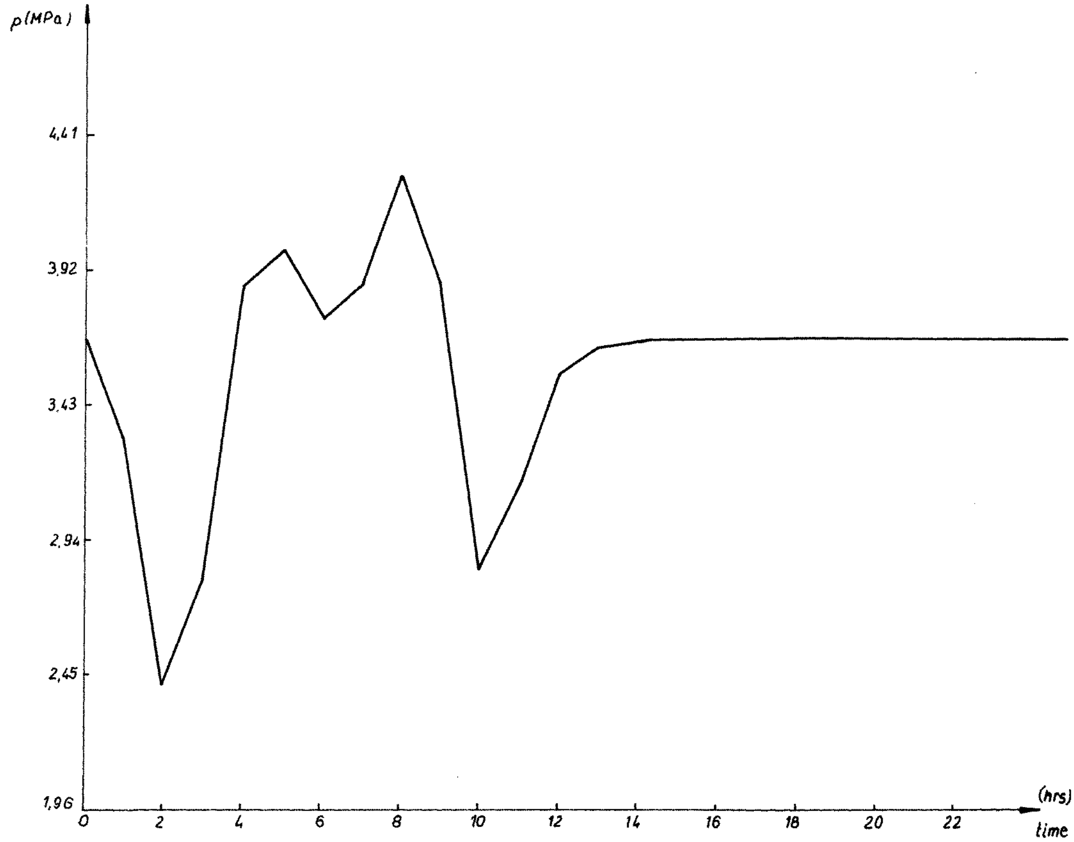


Figure 8. Changes of pressure at $x = L$ obtained using D-F scheme for $L = 7 \times 10^4$ m

through the decomposition of the set of arcs (pipes) of the graph network into subsets of arcs of approximate lengths, attributing various numbers of discretization intervals to individual subsets, This, however, results in a marked increase of complexity of the numerical algorithm.

The implicit schemes Figures 1, 2 and 4 are characterized by high accuracy within a large interval Δt. This allows for a significant reduction of computation time as compared with the explicit scheme in spite of the fact that the new variables P_k^{n+1} are not explicitly defined; however, it is necessary to solve a matrix equation in each time step.

The differences in computation times between the schemes gives in Figures 1, 2 and 4 and the scheme in Figure 3 will increase in step with the dimension of the simulated networks including arcs of highly varied lengths.

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